Outline *In this lecture we will review Myerson's lemma* **?**, *then discuss surplus maximization, knapsack auctions, then talk about the revelation principle.*

Contents

1 Notation

- bidders 1 through n
- v_i valuation of bidder i
- $\mathbf{b}$ bid vector
- $\mathbf{x}(\mathbf{b})$ allocation rule
- $\mathbf{p}(\mathbf{b})$ payment rule
- $u_i(\mathbf{b}) = v_i \cdot x_i(\mathbf{b}) - p_i(\mathbf{b})$ quasilinear utility
- DSIC: dominant-strategy incentive-compatible

there's prob some more stuff, add as needed

2 Myerson's Lemma

Some definitions first:

- Implementable: for a single-parameter environment, an allocation rule $\mathbf{x}$ is implementable if there exists a payment rule $\mathbf{p}$ s.t. the sealed-bid auction $(\mathbf{x}, \mathbf{p})$ is DSIC. In other words, there's a payment scheme that rewards honesty.
- Monotone: for all bidders i and other bids $\mathbf{b}_{-i}$, the allocation $x_i(z, \mathbf{b}_{-i})$ to i is nondecreasing as a function of the bid z. In other words, bidding more means you get more.
- Single-parameter environment: has n bidders, where bidder i has private valuation v_i and feasible set $X = \{(x_1, \ldots, x_n)\}$ of the results of the payment rule – x_i is the amount that bidder i receives.

2.1 Statement

For a single-parameter environment:

(a) An allocation rule $\mathbf{x}$ is implementable if and only if it is monotone.

(b) If $\mathbf{x}$ is monotone, the unique payment rule $\mathbf{p}$ s.t. the sealed-bid auction $(\mathbf{x}, \mathbf{p})$ is DSIC is

$$p_i(b_i, \mathbf{b}_{-i}) = \int_0^{b_i} z \frac{d}{dz} x_i(z, \mathbf{b}_{-i}) dz$$

2.2 Proof

(a) Payment sandwich. Proved by David, but the essence is

$$z[x(y) - x(z)] \leq p(y) - p(z) \leq y[x(y) - s(z)]$$

(b) For piecewise constant, monotone $\mathbf{x}$, the only important bits are the jumps. Because $\mathbf{x}$ is monotonic, we want a $\mathbf{p}$ that only jumps at places z where $\mathbf{x}$ also jumps. So if we look at

1 bidder i, fix everyone else's bids $\mathbf{b}_{-i}$ and look at their allocation $\mathbf{x}$ as a function of their valuation, we get

$$p_i(b_i, \mathbf{b}_{-i}) = \sum_{j=1}^{\ell} z_j \cdot \text{jump in } x_i(\cdot, \mathbf{b}_{-i}) \text{ at } z$$

piecewise constant can approximate any continuous function, so this ends up being

$$p'(z) = zx'(z)$$
$$p_i(b_i, \mathbf{b}_{-i}) = \int_0^{b_i} z \cdot \frac{d}{dz} x_i(z.\mathbf{b}_{-i}) dz$$

3 Surplus Maximization for Knapsack Auctions

Remember that the Vickrey auction was "gold" – it satisfied 3 criteria:

- DSIC
- social surplus-maximizing
- polynomial runtime

Consider another auction setup, called a *knapsack auction*:

- n bidders, each with private valuation v_i and publicly known "weight" w_i
- seller has a capacity W, which **bidders must not exceed** ($\sum_{i=1}^{n} w_i x_i \leq W$, $x_i \in \{0, 1\}$)

so named because it's modeled after the famous knapsack problem – if we have limited space in our knapsack (n items with weights w_i, weight W total) to choose from many valuable things (n items with values v_i), the problem asks which things we should pick to maximize value ($\max \sum_{i=1}^{n} v_i$ where $\sum_{i=1}^{n} \leq W$). Can we make a knapsack auction also "gold"?

No, because the knapsack problem is NP-hard. Thus we can't simultaneously satisfy both of the 2 last "gold" criteria. But if we allow the runtime to balloon up, can we at least satisfy DSIC-ness?

...Well actually in real life we'd like runtime to be polynomial. So something more reasonable to do is make our surplus 'loss' as small as possible, but make sure the algorithm is polynomial time. This is a good deal since it lets us use the more well-studied field of approximation algorithms.

If we know the exact surplus-maximizing $\mathbf{x}$, then by Myerson's lemma, the scheme is also DSIC. But if we relax this to a $\mathbf{x}$ that makes the scheme polynomial-time and approximates the maximum surplus, we can actually still make some pretty good guarantees! In particular if we choose a greedy allocation, we get at least 50% of the theoretical maximum surplus, and amazingly we can get pretty close to this scheme being DSIC.

3.1 The Revelation Principle

DSIC is secretly made of 2 related-but-disparate assumptions:

- Everyone has a dominant strategy regardless of v_i.
- The dominant strategy is to reveal all private information to the mechanism (the auctioneer)

But actually it's possible to have schemes where the dominant strategy doesn't require revealing private information. Consider a malevolent auctioneer who, after seeing everyone's bids $\mathbf{b}$, runs an auction on the bids $2\mathbf{b}$ instead. Then the dominant strategy is for everyone to bid half their value.

Notice, however, that the bidders' new bid was related to their original secret by a factor of 2 – the *revelation principle* makes this more clear, saying that the second requirement follows from the first.

3.1.1 Statement

For every mechanism M where each participant has a dominant strategy, there is an equivalent DSIC mechanism M' where the dominant strategy is to reveal all private info to M'.

3.1.2 Proof

Have M' act as a sort of 'middleman', accepting sealed bids $\mathbf{b}$ from the players, submitting them to M, and choosing the same outcome that M does. Then M' is DSIC, since if anyone submits a bid other than their private information v_i, M' will then submit that other bid. Remember that in M, each participant is using their dominant strategy, which means that M would give a suboptimal (lower or equal utility) outcome.